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THE SCALE FACTORS FOR POWER SPECTRAL ANALYSIS ON THE TIME/DATA 100 COMPUTER

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JULY 1970

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I. Introduction

The Time/Data 100 is a small digital computer specialized in time-series analysis.

Its algorithms include the Direct Fourier Transform, the Auto-Spectral Density and the Autocorrelation.

This report will analyze the Power Spectral Averaging Algorithm [1]. This algorithm applies the Direct Fourier Transform Algorithm to an input record. Then it squares and adds the real and imaginary components of the Direct Fourier Transform and accumulates each successive output record.

An input record is a 1001 word block, where each word has 8 bits. An output record also consists of 1001 words, but has 18 bits per word.

The computer has an oscilloscope and a plotter for analog output and tapes for digital output but they can only display 8 bits of the 18 bit output word.

The way the Direct Fourier Transform Algorithm operates and the limited display range of an output word dictated a need for a scale factor so that the output data could be analyzed properly.

A scale factor was derived by analyzing the process of computing the power spectrum of an input data record. Each part of the scale factor is examined and explained as to why it's needed. The last section will show some examples to verify the above scale factor.

II. Input Data Factor

Fo is the allowable maximum positive value of the input data.

$$Fo = F/(\text{data normalization factor})$$

where $F = 127 = 2^7 - 1$ is the maximum number that the Time/Data 100 can receive in an 8 bit input word.

The data normalization factor is a number that is multiplied to the data before it is used as input to the Time/Data 100. It is used to scale the data so that all of it will be within the $\pm$ 127 range. Note that the Time/Data 100 treats the total range as $\pm$ 1 therefore the data is automatically divided by 127. The factor F arises from this fact. For example, in our case the data values were so small that we could multiply them by 10 to achieve greater accuracy. Therefore $Fo = F \cdot (1/10) = 12.7$

Fo is squared because the operation was completed before execution of the Spectral Averaging Algorithm. Therefore Fo^2 must be multiplied to the output data.

III. Input Scale Factor

N1 is the input scale factor.

The Time/Data 100 outputs an 18 bit word for each point in the Direct Fourier Transform Algorithm. Since the Spectral Averaging Algorithm only uses 7 bits plus the sign bit of the 17 bits plus the sign bit output word for its input, the operator must select a 7 bit range with the amplitude scaling switch that will include the most significant bits of the 18 bit output word. The selected number on the amplitude scaling switch corresponds to N1. This, in effect, divides the original word by a factor of 2^{N1}; N1 being the lowest order bit number of the 18 bit word that the 7 bits were selected from. For example, if the most significant bits were from 2^5 through 2^{11}, then by setting the amplitude scaling switch to 5 that bit range will be chosen for input to the Spectral Averaging Algorithm. So the original word is divided by 2^5.

Since this is also done before the Spectral Averaging Algorithm, the factor is squared. So, 2^{2N1} must be multiplied.

IV. Output Scale Factor

N2 is the output scale factor.

The Spectral Averaging Algorithm also outputs a 17 bits plus the sign bit word, but the analog and digital output devices can only display 7 bits plus the sign bit of a word. So, a dynamic range must also be selected on the amplitude scaling switch to display the most significant bit. As in the input scale factor, this has the effect of dividing the output word by 2^{N2}. So, 2^{N2} must be multiplied to counteract the previous division.

V. Data Shifts

The Time/Data 100 uses an implementation of the Rapid Fourier Transform for the Direct Fourier Transform Algorithm. While transforming the data, some data folds are used. Of these data folds, two are sums and they produce 9 bit output words, 8 bits plus a sign bit. To conform to the multiplier requirements, which can only take 7 bits plus a sign bit, the data must be shifted down one bit. This is done twice, so the data is divided by 2^2. This is also done before the Spectral Averaging Algorithm, so 2^4 must be multiplied to the output data.

VI. Averaging Factor

N' is the number of output records that have been accumulated. The Spectral Averaging Algorithm only adds the output records. So, to obtain an averaged spectrum, the output data must be divided by the number of accumulated records.

VII. Algorithm Implementation Factor

The final form of the scale factor had to be derived by analyzing the Time/Data 100's method of computing the average value of the power spectrum and comparing it to the true value of the power spectrum.

Given a time λ, suppose that there exists a continuous function $F(t + \lambda)$, $-\infty<t<\infty$, that satisfies the ergodic hypothesis and a square window $B(t)$ such that

$$B(t) = 1 \qquad \text{for } |t| \le Tn/2$$

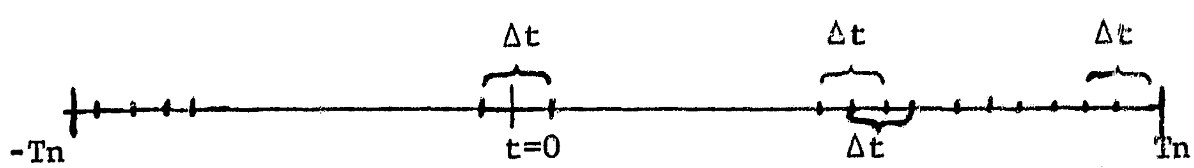
$$B(t) = 0 \qquad \text{for } |t| > Tn/2$$

$$B(t) = B(-t)$$

for some interval of time Tn.

Let $h(t + \lambda) = F(t + \lambda)\, B(t)$, $-Tn \le t \le Tn$. Partition the interval $-Tn \le t \le Tn$ into $4N + 1$ parts, where N is any integer such that $N \ge 0$ and let Δt be the length of one partition such that $(4N + 1)\, \Delta t = 2Tn$. Finally, let the discrete function $g(k)$, $-2N \le k \le 2N$ be the set of points of $h(t)$ such that they are the midpoints of each partition. Figure 1 describes the partitioned interval. Note that the distance between two points of $g(k)$ is also Δt, which is the sampling rate.

Figure 1.



$\longleftarrow \quad (4N + 1)\, \Delta t = 2Tn \quad \longrightarrow$

This midpoint of each partition is a point of $g(k)$.

The Direct Fourier Transform Algorithm assumes that a 1001 point input record is appended by 500 zero points on each side. Hence N is 500 in this algorithm.

So, the Time/Data 100 Computes the Direct Fourier Transform with this approximation:

$$A(j) = \sum_m g(m) \; e^{-i2\pi (j\Delta f)(m\Delta t)} \tag{1}$$

(Throughout this report the variables j, 1, and m vary from -2N to 2N and $\sum$ means $\sum_{-2N}^{+2N}$ unless otherwise specified.)

where Δt is the sampling rate and $\Delta f = 1/2\ Tn = 1/(4N+1)\ \Delta t$

$$A(j) = \sum_m g(m) \; e^{-i2\pi jm/(4N+1)}$$

The Power Spectral Density Algorithm then computes:

$P(j) = A(j) \cdot A^{*}(j)$,

where * represents the complex conjugate.

$$P(j) = \sum_m \sum_r g(m)\; g(r)\; e^{i2\pi j(m-r)/(4N+1)}$$

This approximation, however, doesn't use the sampling rate, Δt, or the length of the input interval, Tn, in the computation of the power spectrum.

Rewrite equation (1):

$$A(j\Delta f) = (1/\Delta t) \sum_m g(m\Delta t)\; e^{-i2\pi (j\Delta f)(m\Delta t)} \Delta t$$

Express the summation as an integral and replace the discrete function $g(m\Delta t)$ by its continuous counterpart $F(t+\lambda)\ B(t)$

$$A\ (f_j,\lambda) = (1/\Delta t) \int F\ (t+\lambda)\ B(t)\ e^{-i2\pi f_j t}\ dt$$

where the integrals have the limits $-\infty$ to ∞; this applies to what follows.

The Power for frequency f_j is:

$$P(f_j,\lambda) = (1/\Delta t)^2 \iint F\ (t+\lambda)\ B(t)\ F(s+\lambda)\ B(s) e^{i2\pi f_j (t-s)} dsdt$$

Ensemble averaging is defined by:

$$P'(f_j) = \lim_{T\to\infty} \frac{1}{2T} \int_{-T}^{T} P(f_j,\lambda)\ d\lambda$$

where $P'(f_j)$ is the average value of $P(f_j,\lambda)$ over all possible records.

$$P'(f_j) = \lim_{T\to\infty} (1/2T) \int_{-T}^{T} (1/\Delta t)^2 \iint F(t+\lambda)\ B(t)\ F(s+\lambda)\ B(s) e^{i2\pi f_j (t-s)} dsdtd\lambda$$

Rearrange Terms:

$$P'(f_j) = (1/\Delta t)^2 \iint B(t)\ B(s) e^{i2\pi f_j (t-s)} \lim_{T\to\infty} (1/2T) \int_{-T}^{T} F(s+\lambda)\ F(t+\lambda)\ d\lambda dsdt$$

Let $\tau = t + \lambda$

$d\tau = d\lambda$ and $s+\lambda = (s-t) + \tau$

$$P'(f_j) = (1/\Delta t)^2 \iint B(t)\, B(s)\, e^{i2\pi f_j(t-s)} \lim_{T\to\infty} (1/2T)\int_{-T}^{T} F[(s-t)+\tau]\, F[\tau] d\tau ds dt \qquad (2)$$

Note that the last expression is the autocovariance of $F(\tau)$.

Let $C(s-t) = \lim_{T\to\infty} (1/2T)\int_{-T}^{T} F[(s-t)+\tau]\, F(\tau)\, d\tau$

The autocovariance function is also defined by: [2]

$$C(\sigma) = \int G(f)\, e^{i2\pi f\sigma} df \qquad (3)$$

where

$$G(f) = \int C(\sigma)\, e^{-i2\pi f\sigma} d\sigma$$

is the power spectral density function.

Substitute (3) into (2)

$$P'(f_j) = (1/\Delta t)^2 \iint B(t)\, B(s)\, e^{i2\pi f_j(t-s)} \int G(f)\, e^{i2\pi f(s-t)} df ds dt$$

$$P'(f_j) = (1/\Delta t^2) \int G(f) df \int B(t) e^{i2\pi t(f_j-f)} dt \int B(s) e^{-i2\pi s(f_j-f)} ds$$

$$P'(f_j) = (1/\Delta t^2) \int G(f)\ H(f_j-f)\ H^*(f_j-f)\, df$$

where $H(f) = \int B(t)\, e^{i2\pi ft} dt$

and * represent the complex conjugate

$$P'(f_j) = (1/\Delta t^2) \int G(f)\, |H(f_j-f)|^2 df \qquad (4)$$

Except for the scale factor, $1/\Delta t^2$, this is the convolution of the power spectral density with a spectral window.

Compute the spectral window,

$$H(f_j-f) = \int B(t)\, e^{i2\pi(f_j-f)t} dt$$

$$= \int_{-Tn/2}^{Tn/2} e^{i2\pi(f_j-f)t} dt$$

$$= \left(e^{i\pi(f_j-f)Tn} - e^{-i\pi(f_j-f)Tn}\right) / i2\pi(f_j-f)$$

$$= Tn(\cos \pi\theta + i \sin \pi\theta - [\cos \pi\theta - i \sin \pi\theta]) / i2\pi\theta$$

where $\theta = (f_j - f)\, Tn$

$$H(f_j - f) = Tn \sin(\pi\theta)/\pi\theta = Tn \operatorname{dif} \theta$$

Rewrite equation (4)

$$\Delta t^2 P'(f_j) = \int G(f)\, Tn^2 (\operatorname{dif} \theta)^2 \, df$$

$$(\Delta t^2 / Tn^2)\, P'(f_j) = \int G(f) (\operatorname{dif} \theta)^2 \, df \qquad (5)$$

The derivation, in effect, stops here. So the spectrum must be interpreted with a spectral window of $(\operatorname{dif} \theta)^2$. The window is pictured in Figure 2.

For the Time/Data 100, we have to replace $P'(f_j)$ with the machines estimate along with the previously defined factors.

$$P'(f_j) = \left(Fo^2 \; 2^{2N1 + N2 + 4}/N'\right) P_{TD}(f_j)$$

where $P_{TD}(f_j)$ is the machine's power spectral estimate.

Rewriting equation (5):

$$[Fo^2 \; 2^{2N1 + N2 + 6} / N' (4N + 1)^2]\, P_{TD}(f_j) = \int_{\infty}^{\infty} G(f) (\operatorname{dif} \theta)^2 \, df \qquad (6)$$

where $\Delta t^2 / Tn^2 = 4/(4N + 1)^2$

This can be simplified if the power has a smoothly varying spectrum and $G(f\ )$ is relatively constant around f_j. Then, we can approximate equation (6) by:

$$\left(Fo^2 \; 2^{2N1 + N2 + 6} / N' (4N + 1)^2 \quad P_{TD}(f_j) \approx G(f_j)\, \delta f\right.$$

where $\delta f = \int (\operatorname{dif} \theta)^2 \, df = 1/Tn$

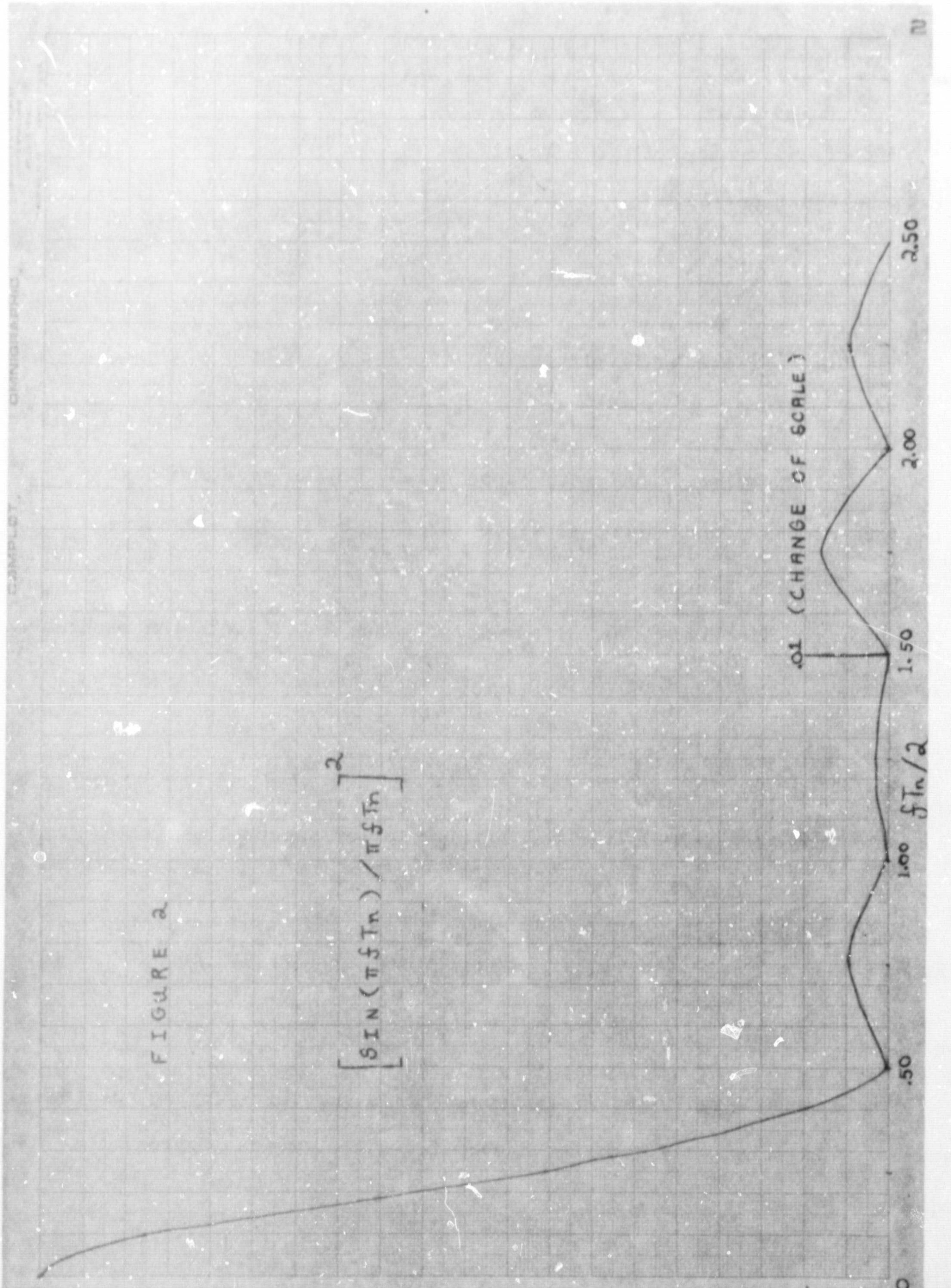


FIGURE 2
[SIN(πfTn)/πfTn]²
fTn/2
.01 (CHANGE OF SCALE)
1.0
.1
0
.50
1.00
1.50
2.00
2.50

$$\left(\left[Fo^{2}\ 2^{2N1 + N2 + 5}\Delta t\right] / N'\ (4N + 1)\right) P_{TD}\ (f_j) \approx G\ (f_j)$$

This estimate is, however, for a double sided spectrum. Therefore, to obtain a single sided spectral estimate, it must be multiplied by 2, namely

$$\left(Fo^{2}\ 2^{2N1 + N2 + 6}\Delta t / N'\ (4N + 1)\right)\ P_{TD}\ (f_j) \approx G\ (f_j).$$

VIII. Examples and Conclusion

To test the scale factor, some examples are taken from the rubidium magnetometer experiment on the OGO-5 satellite that measures the magnitude of the intensity of the magnetic field.

Two methods were used to analyze the test data. First, by using the Time/Data 100 Method and second by using the Blackman-Tukey Method of Power Spectral Analysis.

First, a number of records were accumulated on the Time/Data 100 and the factors were noted. The output was then plotted on the output plotter.

The same data set was then processed by computer with the Blackman-Tukey Method. To compare the two methods, the scale factor was computed and the output from the Blackman-Tukey Method was plotted according to the scale set by the computed scale factor.

To test all portions of the scale factor, the data was sampled at three different rates; this change in Δt brought about a corresponding change in N1, N2, and N' of the scale factor.

First, the data was sampled at 1 point per 144 ms and graphs 1 and 2 show the results of the Time/Data 100 method and the Blackman-Tukey method respectively.

Graphs 3 and 4 show the two sampled at 1 point per 1.008 seconds and 5 and 6 show them sampled at 2.016 seconds.

Upon comparison, these results clearly show that the scale factor does produce the correct results.

Acknowledgement

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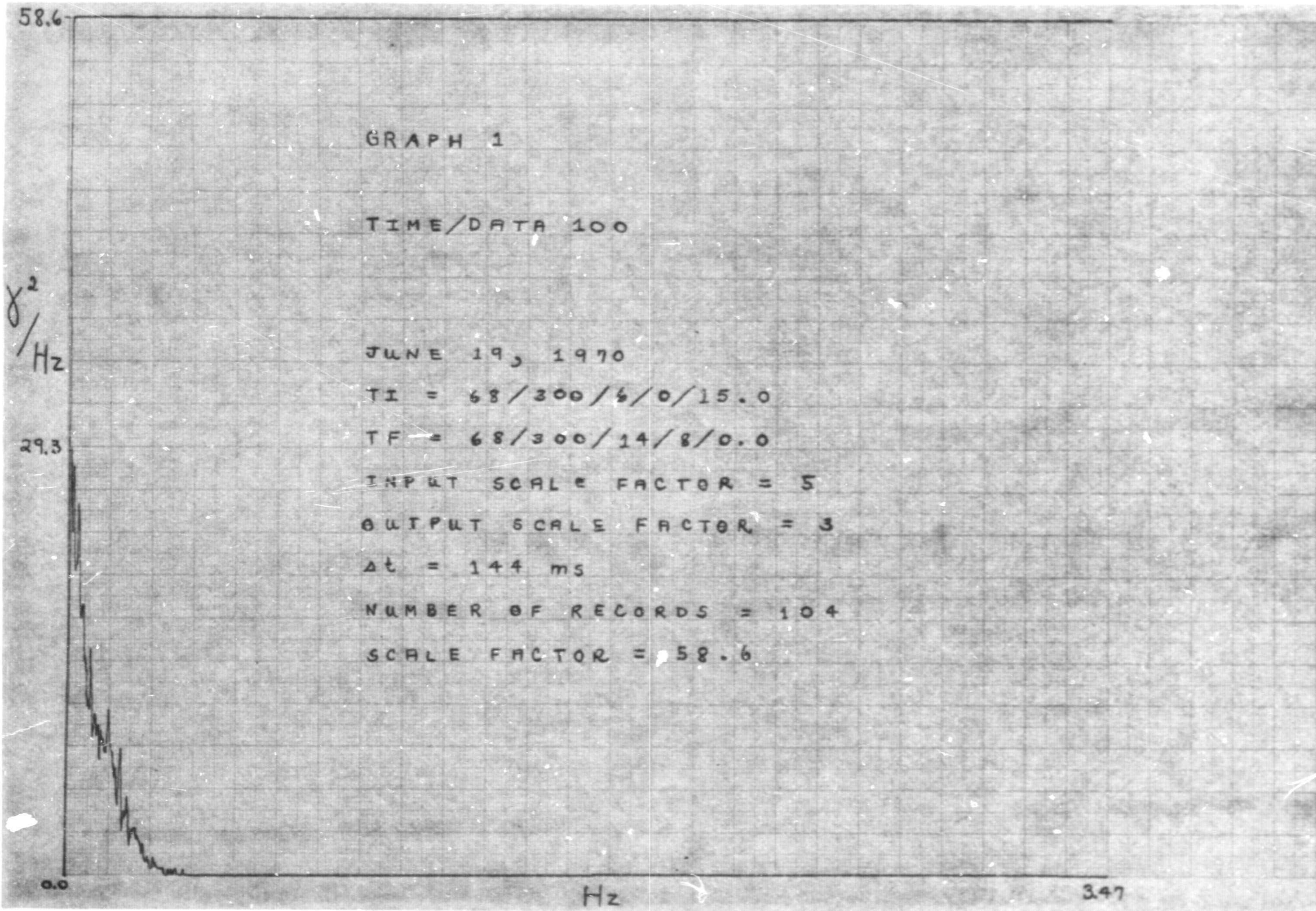
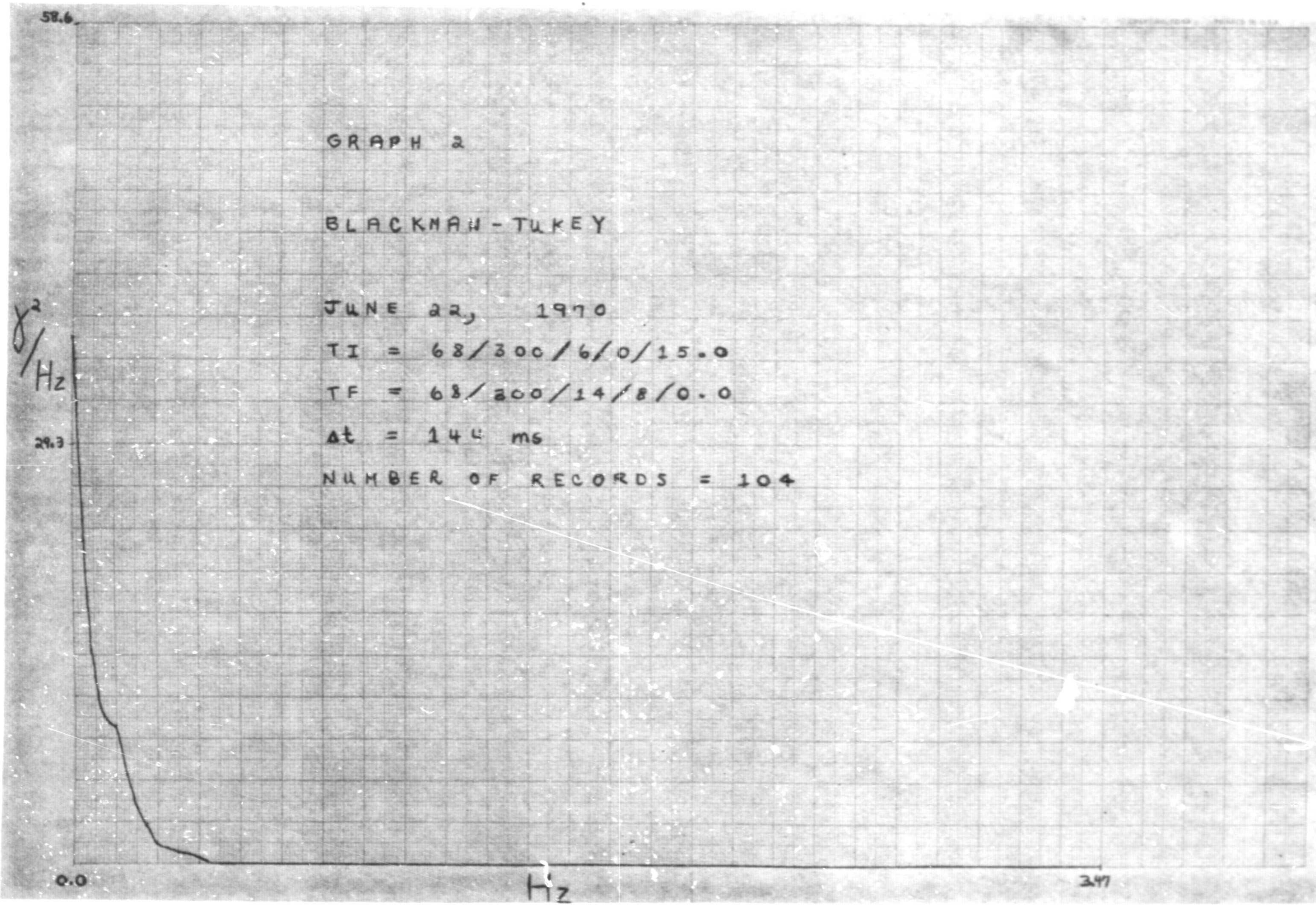
REFERENCES

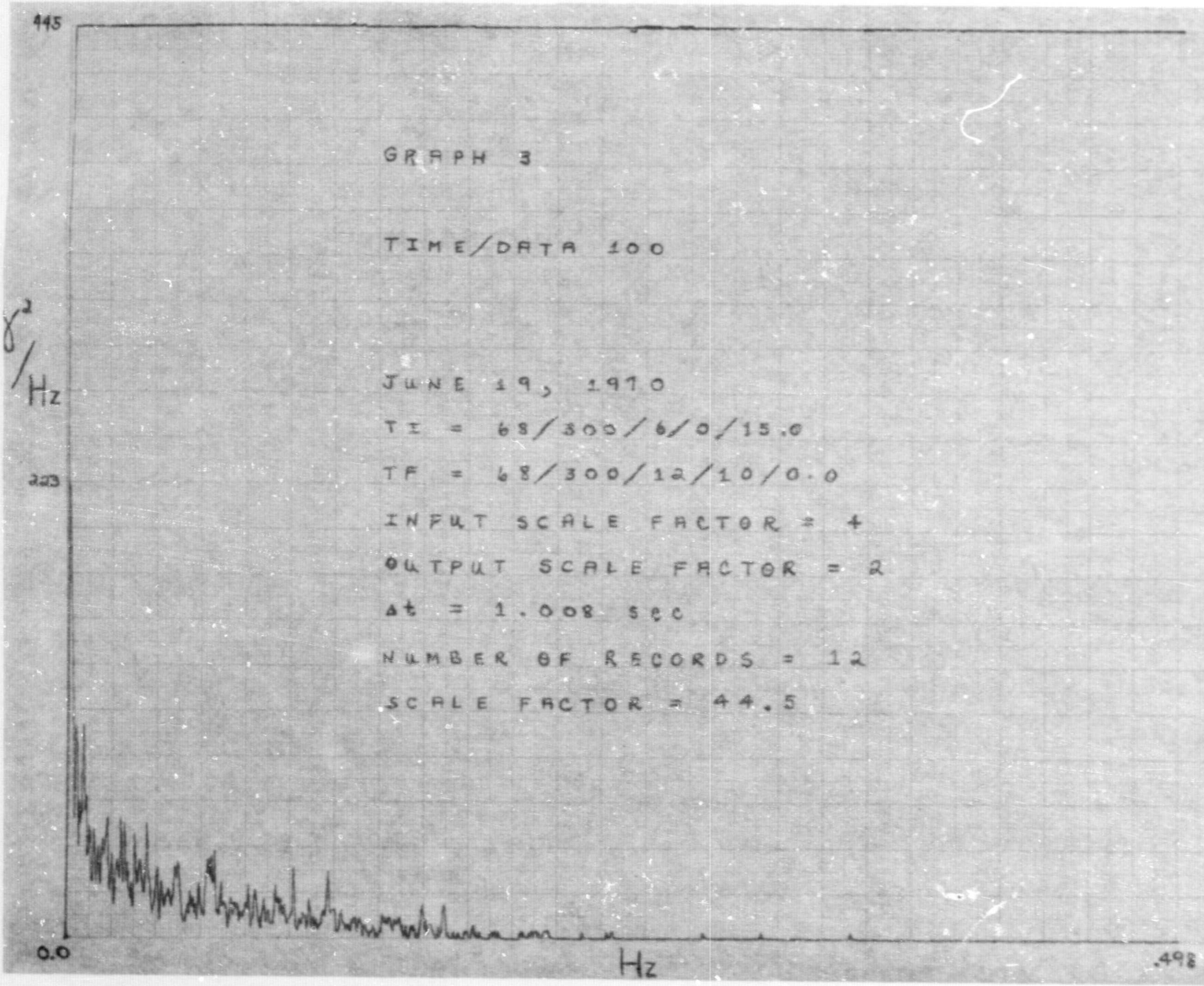
[1]. Time/Data Corporation. "Time/Data 100 Digital Analyzer, Technical Manual," Time/Data Corporation, Palo Alto, California, 1967.

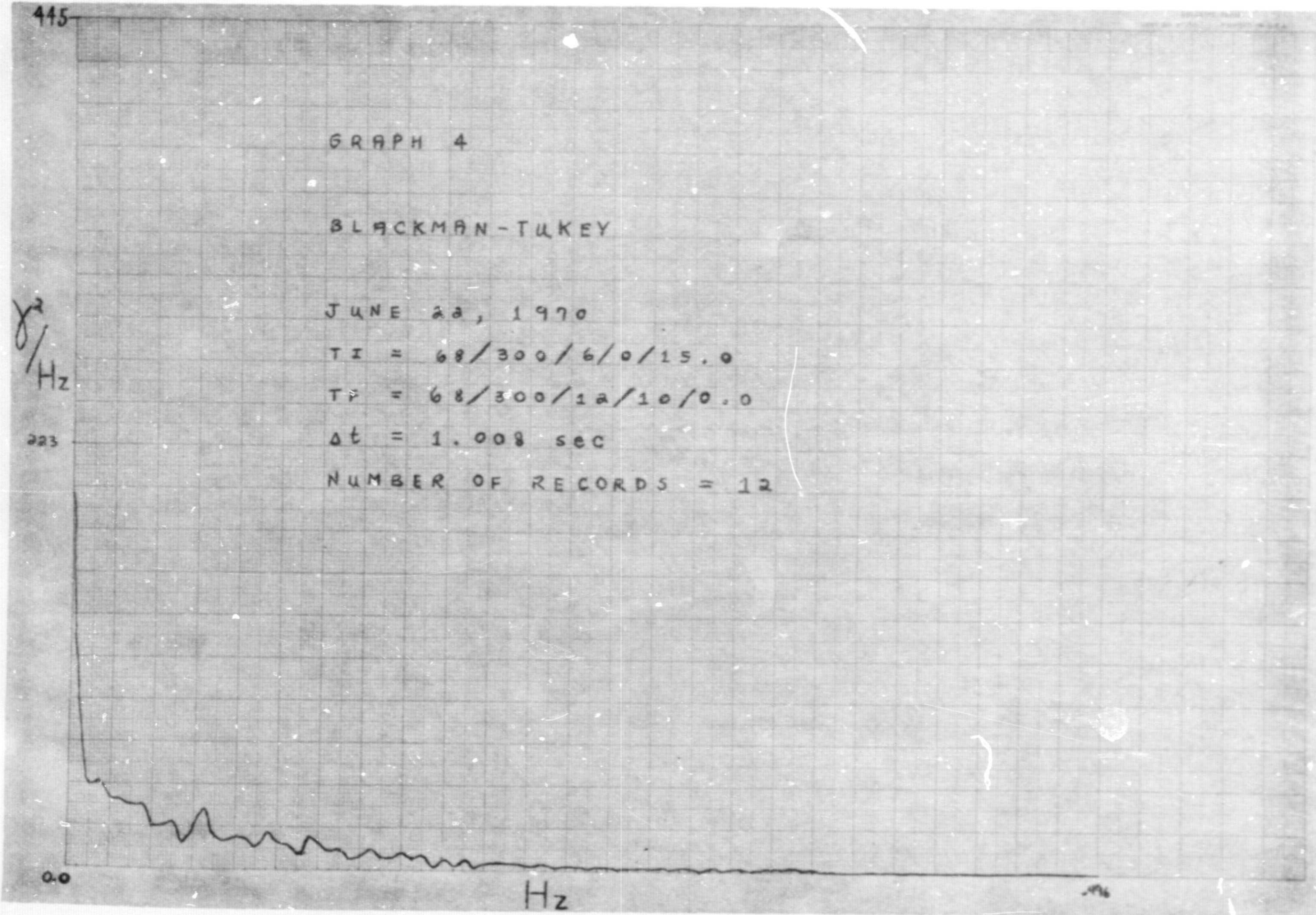
[2]. Blackman, R. B., and Tukey, J. W., "The Measurement of Power Spectra From the Point of View of Communications Engineering," Dover Publications, Inc., New York, 1958.

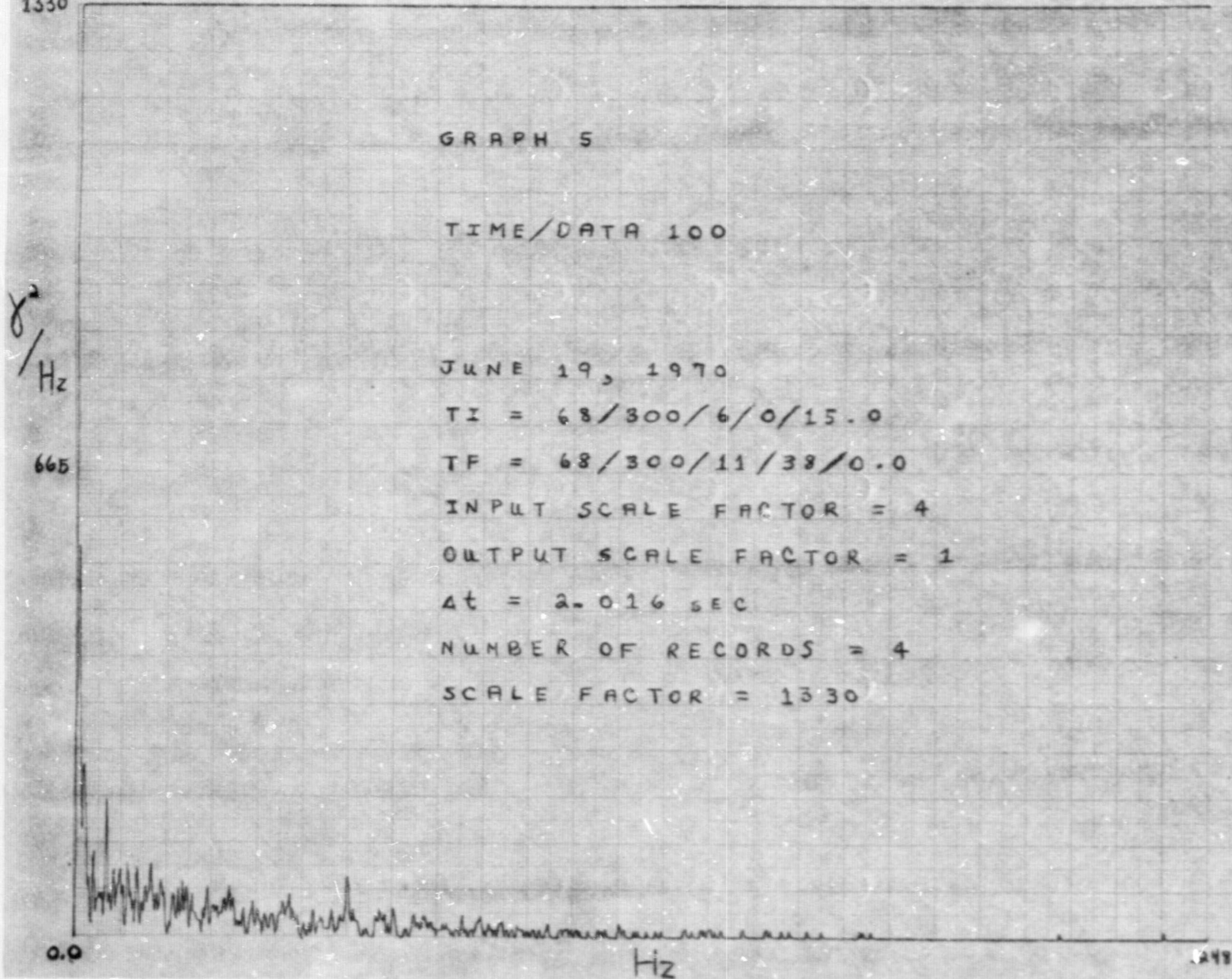

58.6
29.3
0.0
γ^2/Hz
Hz
3.47
GRAPH 1
TIME/DATA 100
JUNE 19, 1970
TI = 68/300/6/0/15.0
TF = 68/300/14/8/0.0
INPUT SCALE FACTOR = 5
OUTPUT SCALE FACTOR = 3
Δt = 144 ms
NUMBER OF RECORDS = 104
SCALE FACTOR = 58.6

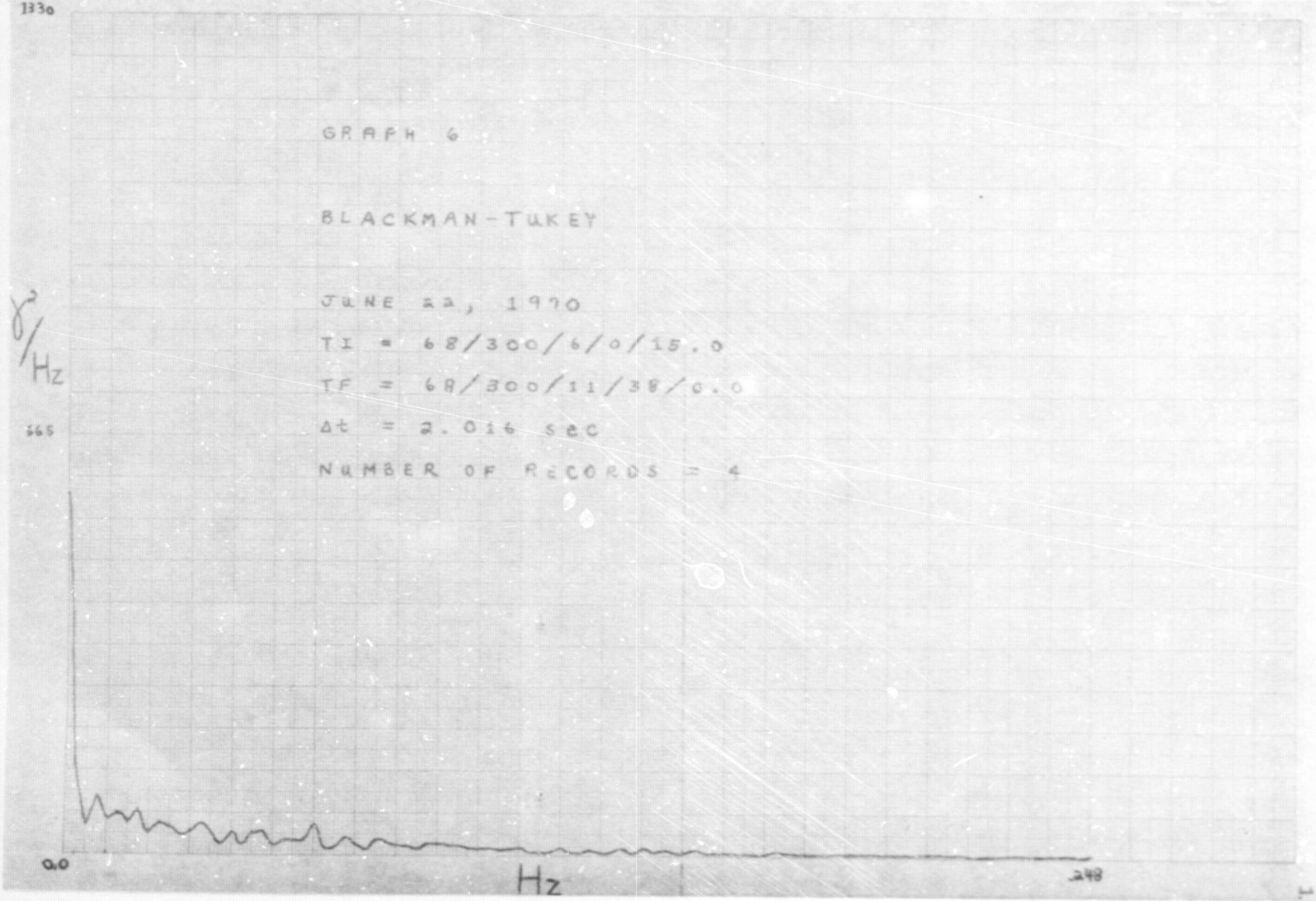


GRAPH 2
BLACKMAN-TUKEY
JUNE 22, 1970
TI = 68/300/6/0/15.0
TF = 68/200/14/8/0.0
Δt = 144 ms
NUMBER OF RECORDS = 104
γ²/Hz
58.6
29.3
0.0
3.47
Hz


445
223
0.0
γ^2/Hz
Hz
.498
GRAPH 3
TIME/DATA 100
JUNE 19, 1970
TI = 68/300/6/0/15.0
TF = 68/300/12/10/0.0
INPUT SCALE FACTOR = 4
OUTPUT SCALE FACTOR = 2
Δt = 1.008 sec
NUMBER OF RECORDS = 12
SCALE FACTOR = 44.5


GRAPH 4
BLACKMAN-TUKEY
JUNE 22, 1970
TI = 68/300/6/0/15.0
TF = 68/300/12/10/0.0
Δt = 1.008 sec
NUMBER OF RECORDS = 12
445
γ^2/Hz
223
0.0
Hz


1330
γ^2/Hz
665
0.0
Hz
.248
GRAPH 5
TIME/DATA 100
JUNE 19, 1970
TI = 68/300/6/0/15.0
TF = 68/300/11/38/0.0
INPUT SCALE FACTOR = 4
OUTPUT SCALE FACTOR = 1
Δt = 2.016 SEC
NUMBER OF RECORDS = 4
SCALE FACTOR = 1330


1330
GRAPH 6
BLACKMAN-TUKEY
JUNE 22, 1970
TI = 68/300/6/0/15.0
TF = 68/300/11/38/0.0
Δt = 2.016 sec
NUMBER OF RECORDS = 4
γ^2/Hz
665
0.0
Hz
.248